FUNDAMENTAL LIMITS FOR DISTRIBUTED LOSSY INTERACTIVE FUNCTION COMPUTATION

Solmaz Torabi, and John MacLaren Walsh

Dept. of Electrical and Computer Engineering
Drexel University

ABSTRACT

In lossy interactive function computation, a series of encoders serially broadcast messages encoded from their observations and previously overheard messages in a round robin fashion, with the aim of enabling a decoder, the central estimation officer (CEO), having access to these messages, to compute a function of all of their local observations up to a given fidelity criterion. Of great interest in this problem is the rate distortion region, describing all possible vectors of rates for which their exist encoders and decoders not exceeding a specified distortion. This paper determines the rate distortion region for several variants of this lossy interactive function computation model in which the observations at the encoders are dependent. First, a rate distortion expression is given for an interactive lossy variant of the Körner Marton problem, in which the function the CEO aims to compute is the binary sum of the observations at each of two encoders observing doubly symmetric binary sources. Building from this result, the rate distortion region for a far more general model in which the CEO is equipped with a side information, and the observations at the encoders need only be conditionally independent given this side information, is determined.

***Index Terms*—** Lossy Compression, function computation, interactive communication, Rate-Distortion

1. INTRODUCTION

With the growing attention to efficient computation for rapid learning from massive datasets and the design of large remote sensing systems, it has become increasingly relevant to understand fundamental limits for distributed function computation. In distributed function computation, a series of terminals encode local observations into messages with the aim of enabling another terminal, the central estimation officer, to compute a function of their collected observations. Multiple variants of this problem have been considered in the literature, and we review only a few of the most closely related information theoretic ones here. In these information theoretic variants, each terminal observes a independently and identically distributed sequence of discrete random variable observations, and the goal is to enable the CEO to compute the function across the observations of different terminals for each element of the sequence. Of particular interest is the region of message rates, measured in bits per sequence element, which enable the function to be computed. This literature can roughly be characterized along three axes: 1) whether or not the encoders must send their messages in one-shot fashion, or if they are allowed to overhear other messages and interact, 2) whether the function must be computed with arbitrarily small probability of error or within an upper bound on average element-wise distortion, and 3) whether or not the observations of different encoders are independent. For non-interactive lossless computation, an important early result [1] established the first fundamental lossless theorem for a point to point case where a function of two sources are reconstructed with one source being uncoded and available to the decoder. This was further generalized to a distributed case [2]. However, the generalization is not yet completely solved for a general function of arbitrary correlated sources [3]. Efforts have been, since then, extended to studying special correlation among the sources [4], and computing special functions [5]. An especially early canonical instance of the one-shot lossless variant of the problem was investigated by Körner and Marton in [5], who provided a method for losslessly computing the binary sum of doubly symmetric binary sources (DSBS) as depicted in Fig. 2 without the need to communicate the sources directly. More broadly, [6] provided inner and outer bound for the rate region of the sources that have neither independence nor symmetric property. If the sources are allowed to interact, building from a rate distortion region for a point to point interactive case [7], the rate region for interactive lossless computation has been obtained in [8] for independent sources. [9] considered this problem in a lossy regime with presence of the side information, which subsumes the model in [8]. This paper investigates a variant of this lossy interactive function computation with dependent observations, and proves the rate distortion region under certain conditions.

Another main source of difficulty after obtaining the rate distortion expression is computing the region, which usually admits cardinality bounds on its auxiliary random variables. If the compressed messages from which the function must be computed, are sent in a one-shot, non-interactive manner, and

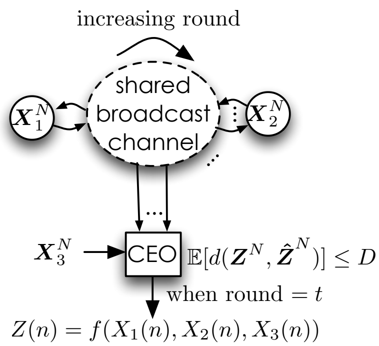

Fig. 1: Three-node lossy interactive function computation

the observations are independnet, [10] admits a Blahut Arimoto type algorithm to iteratively compute the rate distortion region. [8] provides a convex geometric approach algorithm to compute the region for losslessly reproducing a function of distributed independent sources. However, from a practical standpoint, even for small problems, it uses a large series of convex hulls, which is unlikely to be computationally feasible in many contexts. Inspired by this, in addition to the main contribution of this paper, which is a derivation the optimal rate distortion region expression for a new problem, we have included a simplification evaluation of it for binary sources and messages.

After more precisely defining the problem in §2, we begin by finding the rate-distortion region for lossy interactive Körner-Marton type problem. We then explicitly evaluate an achievable region for computing the binary sum of two DSBS (§3). Building on this result, in §4 we consider a harder more general model with presence of the side information, such that given the side information the sources become independent, and a general function of all the sources needs to be computed at the decoder. The main result in the paper shows that the side information can optimally handle the randomness that is common between the sources, therefore, the optimal rate distortion region is can be derived in this conditionally independent case. Finally, an example with binary sources is provided (§4.1).

2. PROBLEM FORMULATION

Consider a network with two source terminals and a single sink terminal as depicted in Fig. 1. Terminal $j = 1, 2$ observes a random sequence $\boldsymbol{X}_j^N = (X_j(1), ..., X_j(N)) \in \mathcal{X}_j^N$. The sink node, the central estimation officer (CEO), observes a side information sequence $X_3(n), n \in [N]$ which is assumed to be correlated with the source variables. The random vectors $X_{1:3}(n) = (X_1(n), X_2(n), X_3(n))$ are iid in time $n \in [N]$, and, $X_1(n) \leftrightarrow X_3(n) \leftrightarrow X_2(n)$. The sink terminal wishes to compute $f : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \to \mathcal{Z}$ in a lossy manner elementwise, estimating the sequence $\boldsymbol{Z}^N = (Z(1), ..., Z(N))$ with $Z(n) = f(X_1(n), X_2(n), X_3(n))$. Let $d : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{Z} \to \mathbb{R}^+$ be a fidelity criterion for this function computation, yielding the block distortion metric

$$d^{(N)}(\boldsymbol{X}_{1:3}^N, \hat{\boldsymbol{Z}}^N) = \frac{1}{N} \sum_{n=1}^{N} d(x_1(n), x_2(n), x_3(n), \hat{z}(n))$$

In order to enable the CEO to estimate the function computation, the nodes take turns broadcasting messages which are perfectly received at both the other nodes and the CEO. The communication is initiated by node 1, and at round i, node $j = ((i-1) \bmod 2) + 1$ sends a message $M_i \in \mathcal{M}_i$, using the encoding function $\psi_i : \mathcal{X}_j^N \times \bigotimes_{k=1}^{i-1} \mathcal{M}_k \to \mathcal{M}_i$ to encode its observations based on the previously overheard messages. After t rounds of communication, the CEO estimates the function sequence based on the messages it has received and the side information using the decoding function $\phi : \mathcal{X}_3^N \times \bigotimes_{k=1}^{t} \mathcal{M}_k \to \mathcal{Z}^N$.

Definition 1. A rate-distortion tuple $(\mathbf{R}, D) = (R_1, R_2, ..., R_t, D)$ is *admissible* for t-message interactive function computation, if $\forall \epsilon > 0$, and $\forall N > n(\epsilon, t)$, there exist encoders $\psi_i, i \in \{1, \ldots, t\}$ and a decoder ϕ with parameters $(t, N, |\mathcal{M}_1|, ..., |\mathcal{M}_t|)$ satisfying $\frac{1}{N} \log_2 |\mathcal{M}_i| \leq R_i \forall i = 1, ..., t$ and $\mathbb{E}\left[d^{(N)}(\boldsymbol{X}_{1:3}^N, \hat{\boldsymbol{Z}}^N)\right] \leq D + \epsilon$ with $\hat{\boldsymbol{Z}}^N = \phi(M_1, \ldots, M_t, \boldsymbol{X}_3^N)$.

Finally, define the collection of admissible rate and distortion vectors $(\mathbf{R}, D)$ to be $\mathcal{RD}^t$. We use this notations again in Section §4, where the complete characterization of the rate region of Fig. 1 is provided.

3. INTERACTIVE LOSSY KÖRNER-MARTON PROBLEM

Consider next a special case of this problem in Fig. 3, inspired by the Körner Marton problem, Fig. 2, in which we have two dependent sources, $X_1 = X \sim \text{Bern}(1/2)$, and $X_2 = X + Z_1$ with $Z_1 \sim \text{Bern}(p_1)$ being independent of X, and the CEO wishes to compute the binary sum of the sources, $X_1 \oplus X_2 = Z_1$ losslessly. For this Körner-Marton problem in 2, it has been shown that to compute this function the set of rate required are $R_1 \geq h(p)$, and $R_2 \geq h(p)$, which improves upon the Slepian-Wolf rate region. This result was among one of the first examples that provided a fundamental limit of compression for computing a particular function without a need to communicate the sources in a non-interactive context. Fig. 3 shows an interactive lossy variant of Körner-Marton problem, where the binary sum should be computed subject to the Hamming distortion. Inspired from [4], since one source is a deterministic function of the other source and the decoder seeks to reproduce a function of the sources (rather than reproducing the sources itself as in [4]), with the function Z_1, being independent of one of the sources, the complete rate-distortion region can be characterized in Theorem 1.

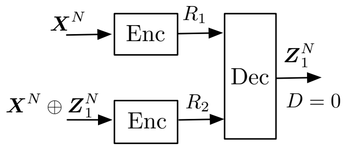

Fig. 2: Körner-Marton Problem

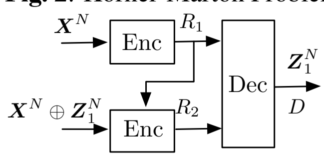

Fig. 3: Lossy Interactive Körner-Marton Problem

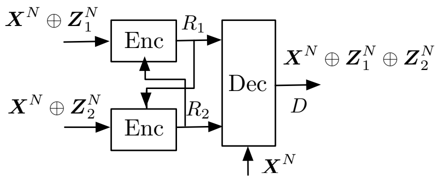

Fig. 4: Lossy Interactive Binary sum of two binary sources with presence of a side information

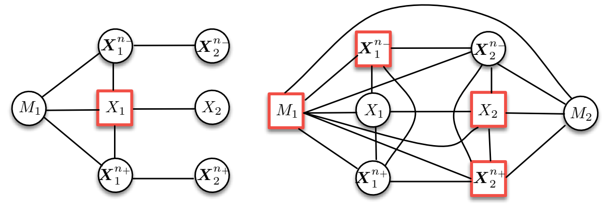

Fig. 5: Markov constraints in Theorem 1. Observed random variables are shown in square.

Theorem 1. *For the model in Fig. 3 the complete characterization of the rate-distortion region can be derived as follows.*

$$\mathcal{RD}_c^2 = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_1 \geq I(X;U_1) \\ R_2 \geq I(X \oplus Z_1; U_2|U_1) \\ \mathbb{E}\big[d(Z_1, \hat{g}(U_{1,2}))\big] \leq D \\ U_1 \leftrightarrow X \leftrightarrow X \oplus Z_1 \\ U_2 \leftrightarrow X \oplus Z_1, U_1 \leftrightarrow X \end{array} \right\} \quad (1)$$

Proof. The achievability proof follows from [9], so we only present the converse proof. We define the auxiliaries $U_1(n) := \{M_1, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}\}, U_2(n) := M_2$. The rate for the first round can be lower bounded as

$$\begin{aligned} R_1 &\geq H(M_1) \geq I(\boldsymbol{X}_1^N, \boldsymbol{X}_2^N; M_1) \\ &= \sum_{n=1}^N H(X_1(n)X_2(n)) - H(X_1(n)X_2(n)|M_1\boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}) \\ &\overset{a}{\geq} \sum_{n=1}^N H(X_1(n)) - H(X_1(n)|M_1\boldsymbol{X}_1^{n-}\boldsymbol{X}_2^{n+}) \\ &= \sum_{n=1}^N I(X_1(n); U_1(n)) \end{aligned} \quad (2)$$

In step a we used the positivity of the conditional mutual information. For the second round of communication:

$$\begin{aligned} R_2 &\geq H(M_2) \\ &\geq I(\boldsymbol{X}_2^N \boldsymbol{X}_1^N; M_2|M_1) \\ &= \sum_{n=1}^N H(X_1(n)X_2(n)|M_1\boldsymbol{X}_2^{n+}\boldsymbol{X}_1^{n-}) \\ &\qquad - H(X_1(n)X_2(n)|M_1M_2\boldsymbol{X}_2^{n+}\boldsymbol{X}_1^{n-}) \\ &\geq \sum_{n=1}^N H(X_2(n)|M_1\boldsymbol{X}_2^{n+}\boldsymbol{X}_1^{n-}) - H(X_2(n)|M_1M_2\boldsymbol{X}_1^{n-}\boldsymbol{X}_2^{n+}) \\ &= \sum_{n=1}^N I(X_2(n); U_2(n)|U_1(n)) \end{aligned} \quad (3)$$

The Markov constraints $M_1, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+} \leftrightarrow X_1(n) \leftrightarrow X_2(n)$, and $M_2 \leftrightarrow X_2(n), M_1, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+} \leftrightarrow X_1(n)$ can be verified in Fig. 5. We use the graphical representation of the factorized distribution to verify the Markov properties, and exploit the conditional independence structure among the random variables. In this undirected graphs, the nodes are random variables appeared in the factorized distribution, and two nodes are connected if they appeared in the same factor. We have $X \leftrightarrow \mathcal{V} \leftrightarrow Y$ if every path between X, and Y contains some node $V \in \mathcal{V}$.

To prove the single letter characterization of the distortion, we use the converse assumption, that there exists a decoding function $\phi(M_1, M_2)$, with nth element ϕ_n, obeying

$$D \geq \frac{1}{N}\sum_{n=1}^N \mathbb{E}[d(Z_1(n), \phi_n(M_1, M_2))] \quad (4)$$

Define the function $g : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{Z}_1^N$ with nth element g_n, to be the Bayes detector for $Z_1(n)$ from M_1, M_2:

$$g_n(M_1, M_2) = \arg\min_{\hat{z} \in \mathcal{Z}_1} \mathbb{E}[d(Z_1(n), \hat{z})|M_1, M_2]. \quad (5)$$

Defining g_n via (5) shows that

$$\mathbb{E}[d(Z_1(n), \phi_n(M_1, M_2))] \geq \mathbb{E}[d(Z_1(n), g_n(M_1, M_2))] \quad (6)$$

Next, define $\tilde{g}_n$ to be the Bayes detector for $Z_1(n)$ from $M_1, M_2, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}$, i.e. let $\tilde{g}_n(M_1, M_2, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}) =$

$$\arg\min_{\hat{z} \in \mathcal{Z}} \mathbb{E}[d(Z_1(n), \hat{z})|M_1, M_2, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}]. \quad (7)$$

The optimality (7) then shows $\mathbb{E}[d(Z_1(n), g_n(M_1, M_2))] \geq \mathbb{E}[d(Z_1(n), \tilde{g}_n(M_1, M_2, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n+}))]$. Proving with (4), (6), that

$$D \geq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[d(Z_1(n), \tilde{g}(U_1(n), U_2(n)))] \tag{8}$$

This proves the converse. □

Here we evaluate and simplify the region for the binary sources with hamming distortion.

Theorem 2. *For the interactive lossy Körner-Marton problem,* $(\boldsymbol{R}, D)$ *is achievable if*

$$R_1 \geq 1 - h(\alpha) \tag{9}$$

$$R_2 \geq h(\alpha * p_1) - h(\beta) \tag{10}$$

for some $0 \leq \beta, \alpha \leq 1/2$ *such that* $\beta * \alpha = D$.

Proof. Let U_1 be the output of BSC(α) with input X. Therefore, we have $X = U_1 \oplus N_1$ where $N_1 \sim$ Bern(α). The second user receiving U_1, generates U_2 which is the output of BSC(β) with input $X \oplus Z_1 \oplus U_1$.

$$X \oplus Z_1 \oplus U_1 = U_2 \oplus N_2 \tag{11}$$

where $N_2 \sim$ Bern(β). We set the decoding function $\hat{g}$ to be $\hat{g}(U_1, U_2) = U_2$. Hence, the distortion achieved at the receiver is

$$\begin{aligned}\mathbb{E}[d(Z_1, \hat{Z}_1)] &= p(Z_1 \oplus \hat{Z}_1 = 1) = p(Z_1 \oplus U_2 = 1) \\ &\overset{a}{=} p(Z_1 \oplus X \oplus Z_1 \oplus U_1 \oplus N_2 = 1) \\ &\overset{b}{=} p(U_1 \oplus N_1 \oplus U_1 \oplus N_2 = 1 = \alpha * \beta\end{aligned}$$

where a follows from (11). In b, we used $X = U_1 \oplus N_1$. Note that, since $\alpha * \beta$ is a monotonically increasing and continues function in both α, and β, all the distortion $0 \leq D \leq 1/2$ can be achieved by this scheme. For $D \geq 1/2$ we can simply let $\hat{g}(U_1, U_2) = 0$. The scheme explained above that achieves distortion $\alpha * \beta$ requires the rate $R_1 \geq I(X; U_1)$ as in (1), which simplifies to $I(X; U_1) = H(X) - H(X|U_1) = 1 - h(\alpha)$. The second rate the simplifies to $I(X \oplus Z_1; U_2|U_1) = H(X \oplus Z_1|U_1) - H(X \oplus Z_1|U_1, U_2) = h(\alpha * p) - h(\beta)$. □

4. THREE-NODE LOSSY INTERACTIVE FUNCTION COMPUTATION

Inspired by the rate distortion region expression for the simple case presented in the previous section, in this section we return to the more complicated model depicted in Fig. 1 in which there is another random variable involved which is provided to the decoder to help the decoder compute a function of all of the sources. For an arbitrary number of users, an inner and outer bound for this problem's rate distortion region was provided in [9], which was not tight in general. In this section, we improve upon the results in [9] by creating a matching outer bound will be proved to be tight for a correlated two-user and a side information scenario, where given the side information, the observations are independent. Note that the model handled in Section §3 is not included in Fig. 1, and Fig. 4. The reason is that in Fig. 3 there is no side information available to the decoder to optimally handle the randomness that is common to X_1, and X_2. The following theorem provides a single letter characterization of the rate-distortion region.

Theorem 3. *For a two-user lossy interactive function computation with a side information* X_3, *where node* $i \in \{1, 2\}$ *observes* X_i, *and* $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, *the rate distortion region is:*

$$\mathcal{RD}_d^t = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_i \geq I(X_1; U_i | U_{1:i-1} X_2) \quad i \text{ odd} \\ R_i \geq I(X_2; U_i | U_{1:i-1} X_1) \quad i \text{ even} \\ j = ((i-1) \bmod 2) + 1 \\ \mathbb{E}\big[d(X_{1:3}, \hat{g}(U_{1:t}, X_3))\big] \leq D \\ U_i \leftrightarrow X_j, U_{1:i-1} \leftrightarrow X_{\{1,2,3\} \setminus \{j\}} \end{array} \right\} \tag{12}$$

where the alphabets $\mathcal{U}_i$ *satisfy* $|\mathcal{U}_i| \leq |\mathcal{X}_j| \prod_{r=1}^{i-1} |\mathcal{U}_r| + 1 + t - i$.

Proof. The achievability proof resembles [9], so we just prove the converse. The lower bound for the rate in the first round can be derived as follows.

$$\begin{aligned} R_1 &\geq H(M_1) \\ &= I(\boldsymbol{X}_1^N \boldsymbol{X}_3^N; M_1 | \boldsymbol{X}_2^N) \\ &= \sum_{n=1}^{N} H(X_1(n) X_3(n) | X_2(n)) - H(X_1(n) X_3(n) | M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N \\ &\geq \sum_{n=1}^{N} H(X_1(n) | X_2(n)) - H(X_1(n) | M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N) \\ &\overset{a_1}{=} \sum_{n=1}^{N} H(X_1(n) | X_2(n)) - H(X_1(n) | M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{\setminus n}, \boldsymbol{X}_2^N) \\ &\geq \sum_{n=1}^{N} H(X_1(n) | X_2(n)) - H(Z_1(n) | M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{\setminus n}, \boldsymbol{X}_2^{n-}, X_2(n)) \\ &\overset{a_2}{=} \sum_{n=1}^{N} I(X_1(n); U_1(n) | X_2(n)) \end{aligned} \tag{13}$$

In a_1 we have $\boldsymbol{X}_3^{n-} \leftrightarrow M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N \leftrightarrow X_1(n)$. See Figure 6 for the proof. In a_2 the auxiliary is chosen to be $U_1(n) := \{M_1, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\setminus n}, \boldsymbol{X}_2^{n-}\}$. For $i \geq 2$, and i even we have

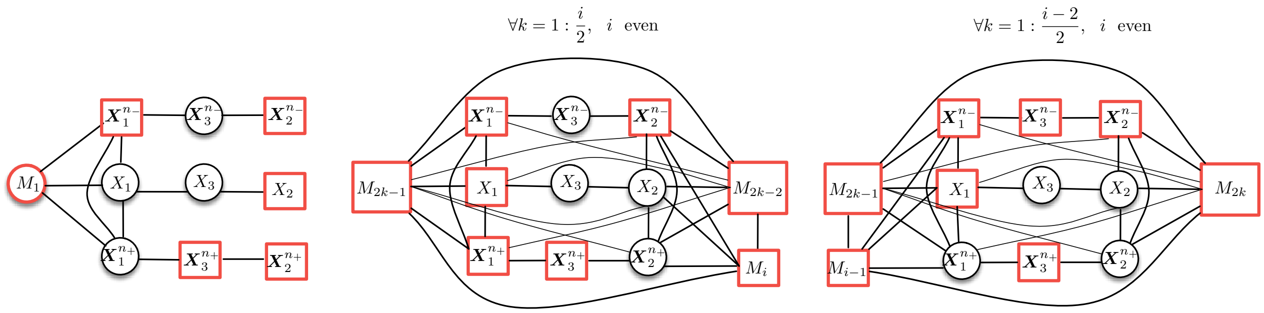

Fig. 6: Markov condition in a_1 (left), b_1 (middle), and b_2 (right) for i even.

$$
\begin{aligned}
R_i &\geq H(M_i) \\
&\geq I(\boldsymbol{X}_2^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_1^N) \\
&= \sum_{n=1}^{N} H(X_2(n)X_3(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) \\
&\qquad - H(X_2(n)X_3(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) \\
&\geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) \\
&\overset{b_1}{\geq} \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\overset{b_2}{=} \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}, X_1(n)) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}, X_1(n)) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\backslash n}, X_1(n)) \\
&\overset{b_3}{=} \sum_{n=1}^{N} I(X_2(n); U_i | U_{1:i-1} X_1(n))
\end{aligned}
$$

For the first term in b_1 we used conditioning reduces the entropy, and for the second term we have $X_2(n) \leftrightarrow M_{1:i}\boldsymbol{X}_1^N\boldsymbol{X}_2^{n-}\boldsymbol{X}_3^{n+} \leftrightarrow \boldsymbol{X}_3^{n-}$. In b_2 we have $X_2(n) \leftrightarrow M_{1:i-1}\boldsymbol{X}_1^{n-}\boldsymbol{X}_2^{n-}\boldsymbol{X}_3^{\backslash n}X_1(n) \leftrightarrow \boldsymbol{X}_1^{n+}$. See Figure 6 for the proof. In b_3 we defined the auxiliaries $U_i := \{M_i\}$ for $i \geq 2$.

For $i \geq 2$, and i odd we have

$$
\begin{aligned}
R_i &\geq H(M_i) \\
&= I(\boldsymbol{X}_1^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_2^N) \\
&= \sum_{n=1}^{N} H(X_1(n)X_3(n)|M_{1:i-1}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{n+}) \\
&\qquad - H(X_1(n)X_3(n)|M_{1:i}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{n+}) \\
&\geq \sum_{n=1}^{N} H(X_1(n))|M_{1:i-1}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{n+}) \\
&\qquad - H(X_1(n)|M_{1:i}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{n+}) \\
&\overset{c_1}{\geq} \sum_{n=1}^{N} H(X_1(n))|M_{1:i-1}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\qquad - H(X_1(n)|M_{1:i}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\overset{c_2}{\geq} \sum_{n=1}^{N} H(X_1(n))|M_{1:i-1}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}, X_2(n)) \\
&\qquad - H(X_1(n)|M_{1:i}, \boldsymbol{X}_2^N, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}) \\
&\geq \sum_{n=1}^{N} H(X_1(n))|M_{1:i-1}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}, X_2(n)) \\
&\qquad - H(X_1(n)|M_{1:i}, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}, X_2(n)) \\
&\overset{c_3}{=} \sum_{n=1}^{N} I(X_1(n); U_i | U_{1:i-1} X_2(n))
\end{aligned}
$$

For the first term in c_1 we used conditioning reduces the entropy, and for the second term we have $X_1(n) \leftrightarrow M_{1:i}\boldsymbol{X}_2^N\boldsymbol{X}_1^{n-}\boldsymbol{X}_3^{n+} \leftrightarrow \boldsymbol{X}_3^{n-}$. In c_2 we have $X_1(n) \leftrightarrow M_{1:i-1}\boldsymbol{X}_3^{n-}\boldsymbol{X}_1^{n-}\boldsymbol{X}_3^{\backslash n}X_2(n) \leftrightarrow \boldsymbol{X}_2^{n+}$. See Figure 7 for the proof of Markov conditions. In c_3 the auxiliaries are chosen to be $U_i := \{M_i\}$ for $i \geq 2$.

With this choice of auxiliaries we can check that the Markov conditions in equation (12) are obeyed. The first Markov condition $U_1(n) \leftrightarrow X_1(n) \leftrightarrow X_2(n), X_3(n)$ can be verified using Figure 8 (left), therefore $M_1, \boldsymbol{X}_1^{n-}, \boldsymbol{X}_3^{\backslash n}, \boldsymbol{X}_2^{n-} \leftrightarrow X_1(n) \leftrightarrow X_2(n)X_3(n)$ holds. For other rounds, the Markov can be verified in Figure 8. □

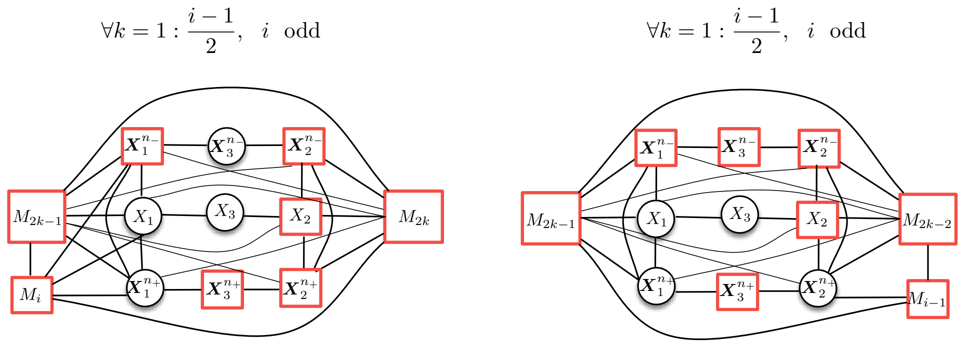

Fig. 7: Markov condition in c_1 (left), and c_2 (right) for i odd

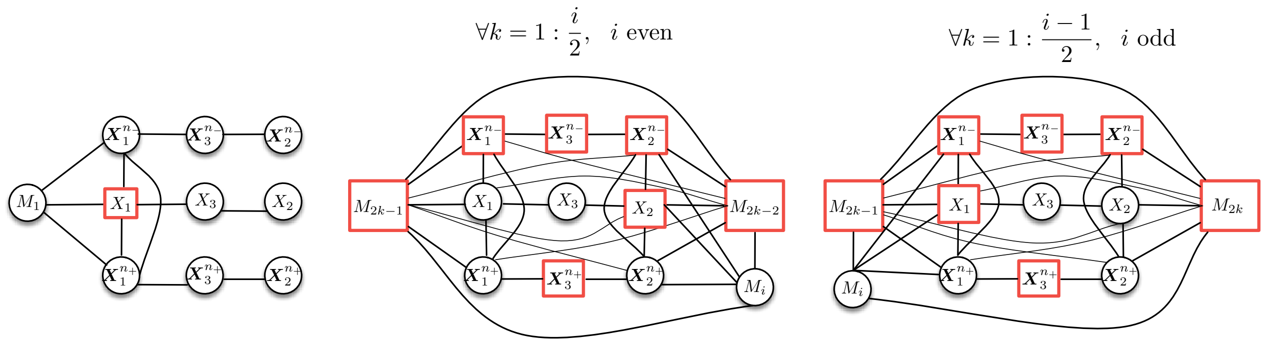

Fig. 8: Markov condition obeyed for the first (left) and i even (middle), and i odd (right) number of rounds

This region has been derived for arbitrary t number of rounds, but, in the next example, we restrict our attention to two number of rounds to discuss the effect of overhearing the first message by the second user before the second message is transmitted. The following lemma is useful in providing the explicit expression for this region.

Lemma 1. *$\mathcal{RD}_d^2$ is a convex region.*

Proof. See section §5 for the proof. □

In the next section, we evaluate and simplify the achievable region for a particular family of message distributions.

4.1. Interactive binary sum computation with presence of the side information

In this section we derive the rate-distortion region for a binary sum computation of discrete symmetric binary sources subject to the Hamming distortion measure. In this case let $\boldsymbol{X}_3^N$ be a sequence of i.i.d. Bernoulli random variables, $X_3 \sim \text{Bern}(\frac{1}{2})$. Let the variables $\boldsymbol{X}_1^N$ and $\boldsymbol{X}_2^N$ be observations of $\boldsymbol{X}_3^N$ through independent binary-symmetric channels with cross-over probabilities p_1, and p_2, respectively. Therefore, $(X_1, X_3) \sim \text{DSBS}(p_1)$, and $(X_2, X_3) \sim \text{DSBS}(p_2)$, and $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, as depicted in Fig. 4. The decoders aims to compute $Z = X_1 \oplus X_2 \oplus X_3$ subject to the hamming distortion. That is, $\mathbb{E}(d(Z, \hat{Z})) = p(Z \neq \hat{Z}) = h^{-1}(H(Z \oplus \hat{Z}))$. Next, we define the following region.

Definition 2. Assume $p_2 \geq p_1$. For each value of D, we define $\eta(D)$ to be set of rates constructed as follows.

$$\eta(D) = \bigcup_{\alpha,\beta,\lambda_1,\lambda_2} \left\{ \mathbf{R} \;\middle|\; \begin{array}{c} R_1 \geq (\lambda_1 + \lambda_2)\left[h(p_2 * p_1 * \alpha) - h(\alpha)\right] \\ R_2 \geq \lambda_1 \left[h(p_2 * p_1 * \beta) - h(\beta)\right] \end{array} \right\}$$

where h is a binary entropy function and the region is defined over all tuple $(\lambda_1, \lambda_2, \alpha, \beta)$, such that $0 \leq \lambda_1, \lambda_2 \leq 1$, and $0 \leq \alpha \leq p_1$, $0 \leq \beta \leq \frac{p_2 - p_1}{1 - 2p_1}$, and

$$D = \lambda_1[\alpha * \beta] + \lambda_2[p_2 * \beta] + (1 - \lambda_1 - \lambda_2)[p_1 * p_2]. \tag{14}$$

Theorem 4. *For computing the binary sum of three doubly symmetric binary sources $(X_1, X_3) \sim$ DSBS(p_1), and $(X_2, X_3) \sim$ DSBS(p_2) subject to the Hamming distortion, we have $\mathcal{RD}_d^2(D) \subseteq \eta^*(D)$*

Proof. To prove the upper bound, we consider a particular joint distribution of the sources X_1, X_2, X_3, and the auxiliary random variables U_1, U_2, and we evaluate the rate distortion expression for this particular distribution. For $D \geq \frac{1}{2}$ we simply let $\hat{g}(U_1, U_2) = 0$. If $D \leq \frac{1}{2}$, then we choose the distributions as follows. First, let (U_1, U_2) be degenerate, such that

$U_1 = U_2 = \emptyset$. Let $\hat{g}(U_1, U_2, X_3) = X_3$. Then, we get distortion $\mathbb{E}\left[d(Z, \hat{g}(U_1, U_2, X_3))\right] = \mathbb{E}\left[d(X_1 \oplus X_2 \oplus X_3, X_3)\right] = p_1 * p_2$. The proof is as follows.

$$\begin{aligned}
&\mathbb{E}\left[d(X_1 \oplus X_2 \oplus X_3, X_3)\right] = p(X_1 \oplus X_2 \oplus X_3 \neq X_3) \\
&= p\left(X_1 \oplus X_2 \oplus X_3 \oplus X_3 = 1\right) = p(X_1 \oplus X_2 = 1) \\
&= p(Z_1 \oplus Z_2 = 1) = p_1 * p_2
\end{aligned}$$

Hence, any distortion $p_1 * p_2 \leq D \leq \frac{1}{2}$ is achievable by this scheme. Second, for $D \leq p_1 * p_2$, let U_1 be the output of a BSC(α), $0 \leq \alpha \leq p_1$, with input X_1 while $U_2 = \emptyset$. We define $\hat{g}(U_1, U_2, X_3) = U_1$. Thus, we get the distortion $\mathbb{E}\left[d(Z, \hat{g}(U_1, U_2, X_3))\right] = \mathbb{E}\left[d(Z, U_1)\right] = \mathbb{E}\left[d(X_1 \oplus X_2 \oplus X_3, U_1)\right] = p_2 * \alpha$. Since $X_1 \oplus X_2 \oplus X_3 \leftrightarrow X_1 \leftrightarrow U_1$, this can be proved using Mrs. Gerber's Lemma

$$\begin{aligned}
&H(X_1 \oplus X_2 \oplus X_3 | U_1) \qquad (15)\\
&= h\Big(h^{-1}\left(H(X_1 \oplus X_2 \oplus X_3 | X_1)\right) * h^{-1}\left(H(X_1 | U_1)\right)\Big).
\end{aligned}$$

To derive the first term in the right hand side of (15), we show that $X_1 \oplus X_2 \oplus X_3$ and X_1 are related via a BSC(p_2).

$$\begin{aligned}
&p(X_1 \oplus X_2 \oplus X_3 \neq X_1) = p(X_1 \oplus X_2 \oplus X_3 \oplus X_1 = 1) \\
&= p(X_2 \oplus X_3 = 1) = p(Z_2 = 1) = p_2 \qquad (16)
\end{aligned}$$

Substituting (16) in (15) we have,

$$H(X_1 \oplus X_2 \oplus X_3 | U_1) = h(p_2 * \alpha) \qquad (17)$$

With this scheme, any distortion $p_2 \leq D < p_1 * p_2$ can be achieved. Hence, the minimum rate for the first user is

$$\begin{aligned}
R_1(p_2 * \alpha) &= I(X_1; U_1 | X_2) = H(U_1 | X_2) - H(U_1 | X_1) \\
&= h(p_1 * p_2 * \alpha) - h(\alpha) \qquad (18)
\end{aligned}$$

Finally, to achieve any distortion in $0 \leq D < p_2$, let U_2 be the output of a BSC(β), $0 \leq \beta \leq \frac{p_2 - p_1}{1 - 2p_1}$, with input $X_2 \oplus U_1$, while U_1 is the output of BSC(α) with input X_1. We define $\hat{g}(U_1, U_2, X_3) = U_2 \oplus X_3$. Thus, the distortion will be $\mathbb{E}\left[d(Z, \hat{g}(U_1, U_2, X_3))\right] = \mathbb{E}\left[d(Z, U_2 \oplus X_3)\right] = \mathbb{E}\left[d(X_1 \oplus X_2 \oplus X_3, U_2 \oplus X_3)\right]$
$= \mathbb{E}\left[d(X_1 \oplus X_2, U_2)\right] = \alpha * \beta$. This can also be seen by Mrs. Gerber's Lemma, since $X_1 \oplus X_2 \leftrightarrow X_2 \oplus U_1 \leftrightarrow U_2$, and

$$\begin{aligned}
&H(X_1 \oplus X_2 \oplus X_3 | U_2) \\
&= h\Big(h^{-1}\left(H(X_1 \oplus X_2 | X_2 \oplus U_1)\right) * h^{-1}\left(H(X_2 \oplus U_1 | U_2)\right)\Big) \\
&\stackrel{a}{=} h(\alpha * \beta) \qquad (19)
\end{aligned}$$

where in a we used the fact that $H(X_1 \oplus X_2 | X_2 \oplus U_1) = H(X_1 | U_1)$ Note that the upper bound for parameter β is derived such that $p_1 * \frac{p_2 - p_1}{1 - 2p_1} = p_2$. As a result, this scheme is achievable for any distortion $0 \leq D \leq p_2$. Therefore, the minimum rate for the second user that can be achieved with the distortion $\alpha * \beta$ is

$$\begin{aligned}
R_2(\alpha * \beta) &= I(X_2; U_2 | U_1 X_1) = H(U_2 | U_1 X_1) - H(U_2 | X_2 U_1) \\
&\stackrel{b}{=} h(p_1 * p_2 * \beta) - h(\beta) \qquad (20)
\end{aligned}$$

Where in b we used the Markov chain, $U_2 \leftrightarrow X_2 U_1 \leftrightarrow U_1 X_1$ and applied Mrs. Gerber's lemma to $H(U_2 | U_1 X_1)$

$$\begin{aligned}
&= h\Big(h^{-1}\left(H(U_2 | X_2, U_1)\right) * h^{-1}\left(H(X_2, U_1 | U_1 X_1)\right)\Big) \\
&= h(p_1 * p_2 * \beta) \qquad (21)
\end{aligned}$$

The remainder of the proof invokes convexity and considers a combination of the three above scenarios.

□

5. PROOF OF LEMMA 1, CONVEXITY

Let D_a and D_b be two distortion values, and let $\{(U_{1,a}, U_{2,a}), \hat{g}_a\}$, and $\{(U_{1,b}, U_{2,b}), \hat{g}_b\}$ be the variables that achieve the point $(\boldsymbol{R}_a, D_a) \in \overline{\mathcal{RD}}^t$, and $(\boldsymbol{R}_b, D_b) \in \overline{\mathcal{RD}}^t$, in the closure of the rate distortion region, respectively. Let Q be an independent time sharing random variable such that $p(Q = a) = \lambda$. Define $U_1 = (Q, U_{1,Q})$, and $U_2 = (Q, U_{2,Q})$, and $\hat{g}(U_1, U_2, X_3) = \hat{g}_Q(U_{1,Q}, U_{2,Q}, X_3)$

$$\begin{aligned}
D &= \mathbb{E}[d(Z, \hat{Z})] = \mathbb{E}[d(Z, \hat{g}(U_1, U_2, X_3)] \\
&= p(Q = a)\mathbb{E}[d(Z, \hat{g}(U_1, U_2, X_3) | Q = a] \\
&\quad + p(Q = b)\mathbb{E}[d(Z, \hat{g}(U_1, U_2, X_3) | Q = b] \\
&= p(Q = a)\mathbb{E}[d(Z, \hat{g}_a(U_{1,a}, U_{2,a}, X_3)] \\
&\quad + p(Q = b)\mathbb{E}[d(Z, \hat{g}_b(U_{1,b}, U_{2,b}, X_3)] \\
&= \lambda D_a + (1 - \lambda) D_b \qquad (22)
\end{aligned}$$

The rate for the first round:

$$\begin{aligned}
&I(X_1; U_1 | X_2) = I(X_1; U_1) - I(X_2; U_1) \\
&= H(X_1) - H(X_1 | Q, U_{1,Q}) - H(X_2) + H(X_2 | Q, U_{1,Q}) \\
&= H(X_1) - \lambda H(X_1 | U_{1,a}) - (1 - \lambda) H(X_1 | U_{1,b}) \\
&\qquad - H(X_2) + \lambda H(X_2 | U_{1,a}) + (1 - \lambda) H(X_2 | U_{1,b}) \\
&= \lambda \left(I(X_1; U_{1,a}) - I(X_2, U_{1,a})\right) \\
&\qquad + (1 - \lambda)\left(I(X_1; U_{1,b}) - I(X_2, U_{1,b})\right) \\
&= \lambda I(X_1; U_{1,a} | X_2) + (1 - \lambda)(I(X_1; U_{1,b} | X_2)) \qquad (23)
\end{aligned}$$

The rate at the second round becomes,

$$\begin{aligned}
&I(X_2; U_2 | U_1 X_1) = H(X_2 | U_1 X_1) - H(X_2 | U_1 U_2 X_1) \\
&= H(X_2 | U_{1,Q}, Q, X_1) - H(X_2 | U_{1,Q}, U_{2,Q}, Q, X_1) \\
&= \lambda H(X_2 | U_{1,a}, X_1) + (1 - \lambda) H(X_2 | U_{1,b}, X_1) \\
&- \lambda H(X_2 | U_{1,a}, U_{2,a}, X_1) + (1 - \lambda) H(X_2 | U_{1,b}, U_{2,b}, X_1) \\
&= \lambda H(X_2 | U_{1,a}, X_1) - \lambda H(X_2 | U_{1,a}, U_{2,a}, X_1) \\
&+ (1 - \lambda) H(X_2 | U_{1,b}, X_1) + (1 - \lambda) H(X_2 | U_{1,b}, U_{2,b}, X_1) \\
&= \lambda I(X_2; U_{2,a} | U_{1,a}, X_1) + (1 - \lambda) I(X_2; U_{2,b} | U_{1,b}, X_1) \qquad (24)
\end{aligned}$$

We define W_1 and W_2 to be the auxiliaries that achieve the the rate tuple $\boldsymbol{R} = (R_1, R_2)$ at the closure of the the rate distortion region for distortion D defined in (22).

$$\begin{aligned} R_1(D) &= I(X_1; W_1|X_2) \\ &\overset{a_1}{\leq} I(X_1; U_1|X_2) \\ &\overset{a_2}{=} \lambda I(X_1; U_{1,a}|X_2) + (1-\lambda)(I(X_1; U_{1,b}|X_2)) \end{aligned} \tag{25}$$

In step a_1, we substitute W_1 with auxiliary U_1, because (U_1, U_2) achieve distortion D but not necessarily achieve the rates in the closure of the rate distortion region (minimum rate). Step a_2 follows from (23).

$$\begin{aligned} R_2(D) &= I(X_2; W_2|W_1 X_1) \\ &\overset{b_1}{\leq} I(X_2; U_2|U_1 X_1) \\ &\overset{b_2}{=} \lambda I(X_2; U_{2,a}|U_{1,a}, X_1) + (1-\lambda) I(X_2; U_{2,b}|U_{1,b}, X_1) \end{aligned} \tag{26}$$

In step b_1 we use the the same argument as in a_1, and step b_2 follows from (24). Recall that $(U_{1,a}, U_{2,a})$ were defined to achieve distortion D_a, and $\boldsymbol{R}(D_a)$, and $(U_{1,b}, U_{2,b})$ were defined to achieve distortion D_b, and rate tuple $\boldsymbol{R}(D_b)$. Therefore, any point that achieves any distortion in between, (D), should have its rate tuple greater than $(R_1(D), R_2(D))$. This proves the convexity of the rate distortion region (12).

6. CONCLUSION

Building upon a newly proven rate distortion region for a lossy interactive variants of the classical Körner Marton problem, this paper determined and presented the rate distortion region for a lossy interactive distributed function computation problem in which the observations being encoded are conditionally independent given a side information at the decoder. Additionally, simple expressions for achievable regions for the special case of computing the sum of doubly symmetric binary sources were presented. Future work will investigate whether these achievable regions match the optimal region dictated by the theorem.

7. REFERENCES

[1] A. Orlitsky and J. R. Roche, "Coding for computing," *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp. 903–917, March 2001.

[2] V. Doshi, D. Shah, M. Medard, and S. Jaggi, "Functional compression through graph coloring," *IEEE Trans. Inform. Theory*, vol. 56, no. 8, pp. 3901–3917, August 2010.

[3] T. S. Han and K. Kobayashi, "A dichotomy of functions f(x,y) of correlated sources (x,y) from the viewpoint of the achievable rate region," *IEEE Trans. Inform. Theory*, vol. 33, no. 1, pp. 69–76, January 1987.

[4] A. B. Wagner, B. G. Kelly, and Y. Altug, "Distributed rate-distortion with common components," *IEEE Transactions on Information Theory*, vol. 57, no. 7, pp. 4035–4057, 2011.

[5] J. Korner and K. Marton, "How to encode the modulo-two sum of binary sources (corresp.)," *IEEE Trans. Inform. Theory*, vol. 25, no. 2, pp. 219–221, 1979.

[6] M. Sefidgaran, A. Gohari, and M. R. Aref, "On körner-marton's sum modulo two problem," in *Communication and Information Theory (IWCIT), 2015 Iran Workshop on*. IEEE, 2015, pp. 1–6.

[7] A. H. Kaspi, "Two-way source coding with a fidelity criterion," *IEEE Trans. Inform. Theory*, vol. 31, no. 6, p. 735, November 1985.

[8] N. Ma, P. Ishwar, and P. Gupta, "Interactive source coding for function computation in collocated networks," *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4289–4305, 2012.

[9] S. Torabi and J. M. Walsh, "Distributed lossy interactive function computation," in *54th Allerton Conf. Comm., Control, Comp.*, To Appear. [Online]. Available: http://www.ece.drexel.edu/walsh/TorabiAllerton16.pdf

[10] G. Ku, J. Ren, and J. M. Walsh, "Computing the rate distortion region for the ceo problem with independent sources," *IEEE Trans. Sig. Proc.*, vol. 63, no. 3, pp. 567–575, Feb. 2015.